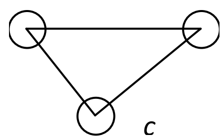
Ballistics and Load Comparisons

Park Hays, 12/13/2009



Suppose you shoot a three shot pattern, shots *a, b,* and *c.* Each shot's position is a random variable, described by a normal distribution. That is, the position of *a* is (X, Y) and $X \sim \mathrm{N}(0, \sigma_{xy}^2), Y \sim \mathrm{N}(0, \sigma_{xy}^2)$.

We are interested in the performance of the classic ballistics measure "maximum spread"—the distance between the furthest two points. To get to maximum spread we first find the statistics of the length between any two shots. The distance between *a* and *b* is

$$D_{ab} = \sqrt{\Delta X^2 + \Delta Y^2} = \sqrt{(X_a - X_b)^2 + (Y_a - Y_b)^2}.$$

The difference between two normally distributed random variables is also a normally distributed random variable, as Wolfram discusses. If the two random variables both have variance σ^2, then the variance of the difference is $2\sigma^2$. In other words, $\Delta X \sim \mathrm{N}(0, 2\sigma_{xy}^2)$, and similarly for ΔY. The savvy reader will recognize that D_{ab} is Rayleigh distributed with shape parameter $\sqrt{2}\sigma_{xy}$.

To be a little bit more explicit, D_{ab} has probability density function $f(x)$ and cumulative distribution function $F(x)$. Each shot is independent, and the distance between each shot pair is independent. If there are n_{shots} then the number of shot pairs is $n_{pairs} = n_{shots}(n_{shots} - 1)/2$. This will be essential later.

The probability that the distance between shot *a* and *b*, D_{ab}, is length x is $f(x)$. The probability that D_{ab} is the maximum spread in the three-shot group is the probability that

- $D_{ab} = x$ and
- $D_{ac} \le x$ and
- $D_{bc} \le x$

Mathematically this is $f(x)F(x)F(x)$. The same maximum value is achieved if D_{bc} or D_{ac} are the maximum, so we multiply the probability by the number of pairs—in this case 3.

Putting this all together the probability density function of the maximum spread is

$$f_m(x) = n_{pairs} f(x) F^{n_{pairs} - 1}(x)$$

Comparing Two Groups' Maximum Spreads

The interesting part comes from comparing one load to another. Typically a few shots of each load are tried—five is common. Then, the handloader compares the maximum spread (or other statistic) from each group, and declares one better than the other. In the next steps I find the probability that the assertion will be wrong. To rephrase, suppose load *A* has underlying standard deviation of σ_{Axy}, and load *B* has σ_{Bxy}, and $\sigma_{Axy} < \sigma_{Bxy}$, (that is, load *A* is better than *B*), what is the probability that the handloader will erroneously declare load *B* to be better than load *A*?

To find this answer we need the probability density function of the difference between two random variables. In this case the two random variables are the maximum spread from group *A* and from group *B*. I use the convention that $\sigma_{Axy} < \sigma_{Bxy}$. Then we are in error if the max spread of *B* is less than the max spread of *A*. Let M_A be the maximum spread from group *A*, and let M_B be the maximum spread from group *B*. Both of these two random variables has a distribution described by $f_m(x)$, but with different shape parameters. For convenience $M_A \sim f_A(x) = f_m(x;\ \sigma_{Axy})$, and $M_B \sim f_B(x) = f_m(x;\ \sigma_{Bxy})$. The difference, $M_B - M_A$ is distributed according the cross-correlation between the two density functions, or

$$M_B - M_A \sim f_{BA}(x) = \int_{-\infty}^{\infty} f_A(t) f_B(x+t) dt$$

Put all the previous steps together, one after another,

$$f_{BA}(x) = n_{pairs}^2 \int_{-\infty}^{\infty} f(x;\ \sqrt{2}\sigma_{Axy}) F^{n_{pairs}-1}(x;\ \sqrt{2}\sigma_{Axy}) f(x+t;\ \sqrt{2}\sigma_{Bxy}) F^{n_{pairs}-1}(x+t;\ \sqrt{2}\sigma_{Bxy}) dt$$

The $f(x)$ and $F(x)$ in the integrand are Rayleigh distributions. For completeness,

$$f(x;\sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{u}(x)$$

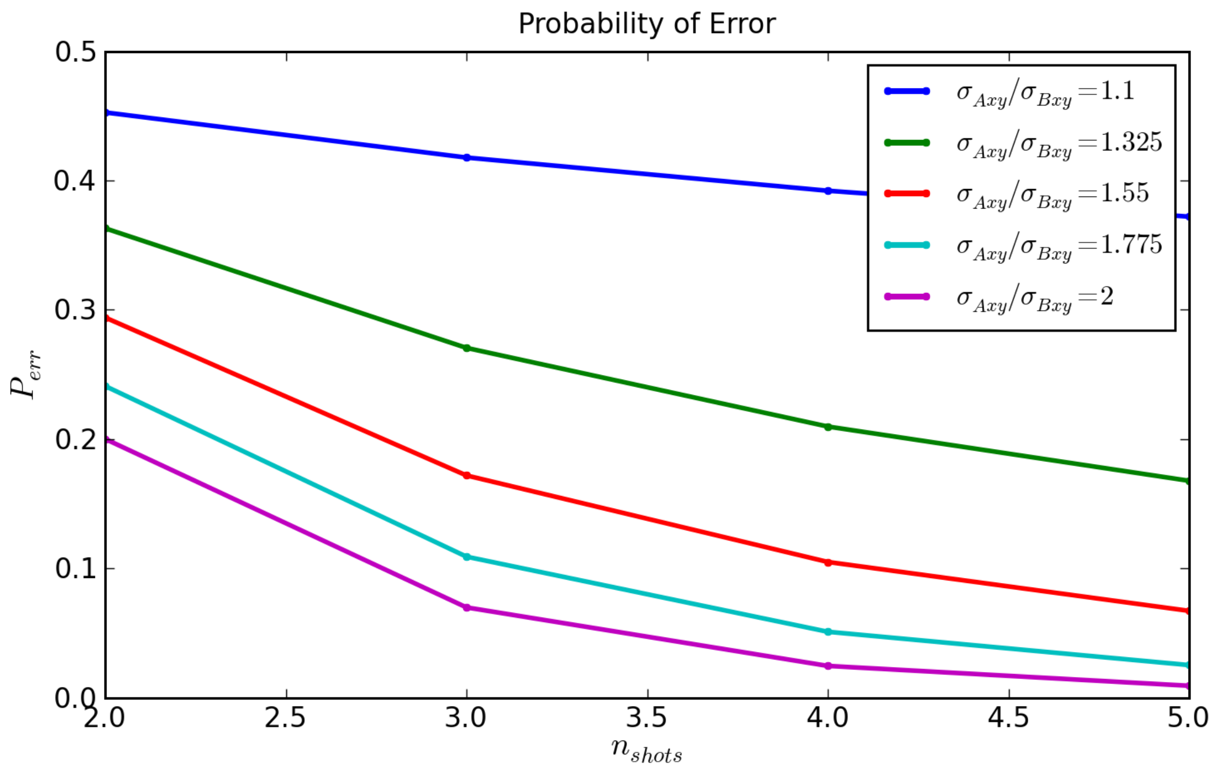
$$F(x;\sigma) = \left(1 - \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) \mathrm{u}(x).$$

I was not able to integrate this analytically, and neither was Mathematica. However, $f_{BA}(x)$ can be evaluated numerically. A more interesting result is the probability of error—that is the probability $M_A > M_B$. This is found by integrated the regions of the density function corresponding to an error, namely the negative values:

$$P_{err} = \int_{-\infty}^{0} f_{BA}(x) dx$$

Including the definition of f_{BA}, we get

$$P_{err} = n_{pairs}^2 \int_{-\infty}^{0} \int_{-\infty}^{\infty} f(x;\ \sigma_A) F^{n_{pairs}-1}(x;\ \sigma_A) f(x+t;\ \sigma_B) F^{n_{pairs}-1}(x+t;\ \sigma_B) dt dx$$



The plots of the numerical solutions show the expected behavior, more shots reduces the chance of error, and increasing contrast ratios also reduces the chance of error. For minor differences, those that the very long range shooters or the benchrest communities might care about, 5 shots produces only about a 73% chance of being correct. Many, many shots may be required to confidently prefer one load to another.

The contrast ratio, $\sigma_{Bxy}/\sigma_{Axy}$, is always known. In the field there are only samples. Future analysis will discuss the confidence with which a handloader could make a determination.